Enrich, X(I) : 30 - 34, July – Dec, 2018

Directed Out-Tree Decomposition of Digraphs

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Abstract

In this paper we define the parameter $\gamma_T^+(D)$, the minimum number of arc-disjoint directed out-trees of the digraph D all with the root (not necessarily same) and we find $\gamma_T^+(D)$ for some standard digraphs.

AMS subject classification: 05C70.

Key words: Out-tree; directed out-tree cover; minimum directed out-tree cover.

1. Introduction

Let G be a digraph with vertex set V(G) and arc set A(G). The out-degree of a vertex v ∈V(G) is denoted by $d^+(v)$, and its in-degree by $d^-(v)$. The maximum in-degree of G is $\Delta^-(G) = \max\{d_G^-(x): x \in V(G)\}$. A digraph is said to be balanced if for every vertex v_i the in-degree equals the out-degree. A balanced digraph is said to be regular if every vertex has the same in-degree and out-degree as every other vertex.

For a graph G, G^* denotes the symmetric digraph of G; that is, the digraph obtained from G by replacing each of its edge by a symmetric pair of arcs.

The concept of arc-disjoint in- and out- branching was introduced by Jorgen Bang-Jensen and Gregory Gutin [2]. Definitions which are not seen here can be found in [1] or [3].

Definition 1.1.[3] Let G = (V, A) be a digraph. A digraph G is said to be an out-tree if

(i) G containsno circuit neither directed nor semi-circuit.

(ii) In G there is precisely one vertex v of zero in-degree. This vertex v is called the root of the out-tree.

Definition 1.2. An out-star (in-star) is an out-tree (in-tree) in which all the vertices are dominated by a root.

Theorem 1.3.[3] An out-tree is a tree in which every vertex other than the root has an in-degree of exactly one.

For convenience, we call the out-tree as directed tree or directed rooted tree throughout this paper. The above definition motivates us to define the directed tree cover of a digraph.

2. Main Results

Definition 2.1. Let D = (V, A) be a digraph. Let ψ be a collection of directed out-trees of D satisfying the following conditions

(i) Every directed out-tree in ψ has at least one arc.

(ii) Every arc of D is in exactly one directed out-tree of ψ.

Then ψ is called a directed out-tree cover of D.

Define $\tau^+(D) = \min\{|\psi| : \psi \in \boldsymbol{D}(D)\}$

where***D***(D) denotes the set of all directed out-tree covers of D. Since A(D) is itself a directed out-tree cover, we have ***D***(D) is non-empty.

Thus, τ^{+}(D) is the minimum number of directed trees in D covering all the arcs of D. If there is no possibility of confusion, we write τ^{+} instead of τ^{+}(D).

A directed out-tree cover ψ of D is called the minimum directed out-tree cover of D if |ψ| = τ^{+}.

Example 2.2.

Consider the digraph D given in the Fig.1

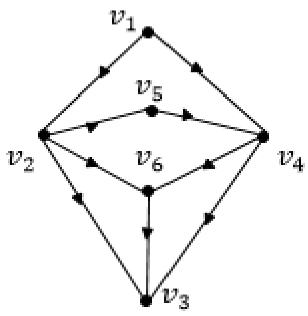

Fig. 1

Let T_1 , T_2 and T_3be the directed out-trees of the digraph D which are shown in Fig. 2.
Clearly, ψ = {T_1,T_2,T_3} is a directed out-tree cover of D. Also it is a minimum directed out-tree cover. Hence, τ^{+}(D) = 3.

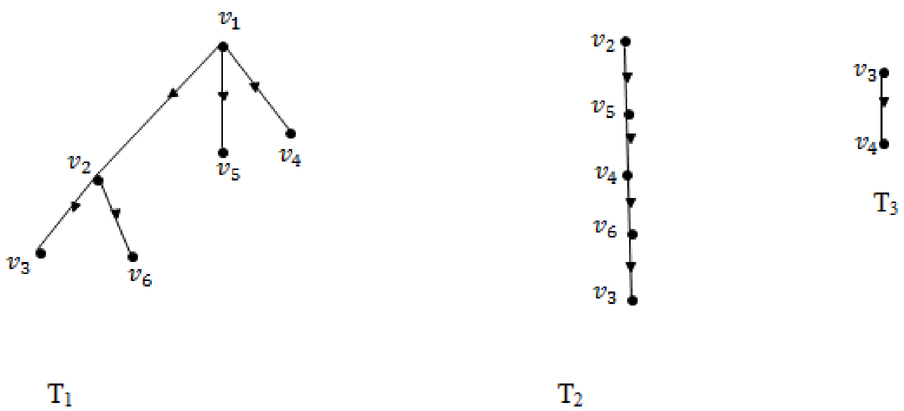

Fig. 2

Theorem 2.3. For any digraph D, $\tau^{+}(D) \geq \Delta^{-}(D)$. In particular for any regular digraph D, $\tau^{+}(D) \geq \Delta^{-}(D)+1$.

Proof: Since any vertex in a directed out-tree is of in-degree at most 1, the number of directed out-trees in a minimum directed out-tree cover is at least $\Delta^{-}(D)$. Thus, we have $\tau^{+}(D) \geq \Delta^{-}(D)$. However, in a regular digraph $\tau^{+}(D) \geq \Delta^{-}(D)+1$, as every vertex has the same in-degree and also the root vertex has in-degree zero.

Theorem 2.4. For any complete symmetric digraph, $\tau^{+}(K_n^*) = n$.

Proof: Since K_n^* is a (n-1)-regular digraph, we have $\tau^{+}(K_n^*) \geq n-1+1 = n$ by Theorem 2.3. Let $V(K_n^*) = \{v_1, v_2, \ldots, v_n\}$. First let n be even, say n = 2k. Fori = 1,2,…..,k, let T_i be the directed out-tree as shown in Fig. 3. In T_i, v_i is adjacent to v_{i+1}, v_{i+2}, ……,v_{i+k-1}, v_{i+k} and

v_{i+k} is adjacent to v_{i+k+1}, v_{i+k+2}, ….., v_{i+2k-1} (subscripts modulo n) and let T_i' be the directed out-tree as shown in Fig. 4.

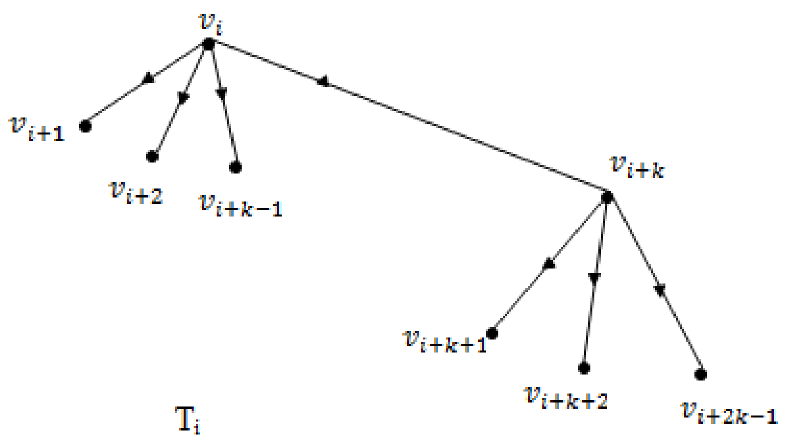

Fig. 3

In T_i', v_{i+k} is adjacent to v_{i+k-1},…..,v_{i+1}, v_i and v_i is adjacent to v_{i+k+1}, v_{i+k+2}, ….., v_{i+2k-1}.

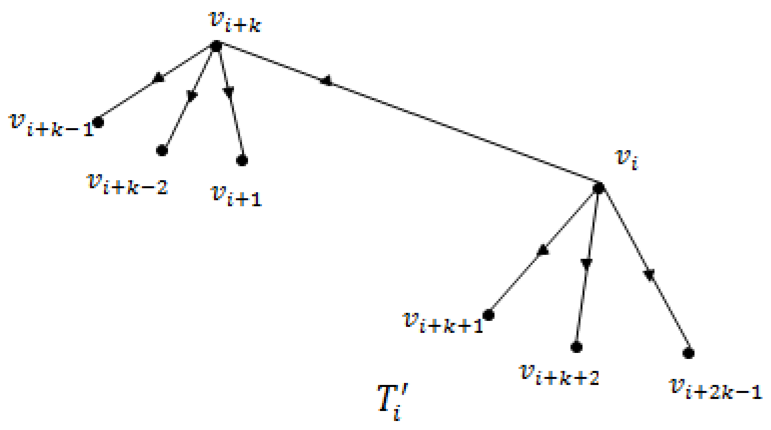

Fig. 4

Clearly, $\psi = \{T_1, T_2, \ldots, T_k, T_1', T_2', \ldots, T_k'\}$ is a directed out-tree cover for K_n^*. This implies, $\tau^+(K_n^*) \le k+k=2k=n$. Thus, $\tau^+(K_n^*)= n$ if n is even. Here we note that each T_i or T_i' in the above directed out-tree cover ψ is a directed spanning out-tree.

For n odd, say n = 2k+1. We have 2k+2 = 2(k+1) spanning directed out-trees which form a directed out-tree cover for K^*_{n+1}. Fix a vertex v_{2k+2} and delete it from each out-tree in the above directed out-tree decomposition of K^*_{n+1}. Leaving out the isolated vertices, we get a directed out-tree cover for K_n^*. This cover contains 2k+2 rooted trees and in which two trees are rooted with the vertex v_{k+1}. Identify those two trees at the vertex v_{k+1} and hence we get 2k+1 = n spanning directed out-tree decomposition of K_n^*. Hence, $\tau^+(K_n^*)= n$, if n is odd.

Theorem 2.5. For an alternating wheel W_n($n \ge 4$ and n is even), $\tau^+(W_n) = \frac{n}{2}$.

Proof: Define the alternating wheel $W_n = (V, A)$ where n is even, $V = \{c, v_1, v_2, \ldots, v_n\}$ and
$A = \{(c, v_i) : 1 \le i \le n \text{ and } i \equiv 1 (\text{mod}2)\} \cup$
$\{(v_i, c) : 1 \le i \le n \text{ and } i \equiv 0 (\text{mod}2)\} \cup \{(v_1,v_n), (v_1,v_2)\} \cup$
$\{(v_i,v_{i+1}) : 3 \le i \le n-1 \text{ and } i \equiv 1 (\text{mod}2)\} \cup$
$\{(v_i,v_{i-1}) : 3 \le i \le n-1 \text{ and } i \equiv 1 (\text{mod}2)\}$

By Theorem 2.3, we have $\tau^+(W_n) \ge \frac{n}{2}$. It is enough to prove the reverse inequality. First, we consider the case for n = 4.

Let $T_1 = \langle v_1,v_4,c,v_3\rangle \cup \langle v_1, v_2\rangle$ and $T_2 = \langle v_3,v_2,c,v_1\rangle \cup \langle v_3, v_4\rangle$.

The directed out-trees $\{T_1,T_2\}$ form a directed out-tree cover for W_n and hence $\tau^+(W_n) \le 2 = \frac{n}{2}$. Thus, the result follows for the case n = 4.

For n > 4, we construct the directed out-trees for W_n as follows:

Let $1 \le i \le n$, $i \equiv 0$ (mod2) and $i \ne 4$, n.

$T_{i/2} = \langle v_{i+1}, v_i, c\rangle$

When i = 4, $T_{i/2} = \{(v_{i+1},v_i,c,v_1,v_2),(v_5,v_6)\} \cup \bigcup_{r=3}^{\left(\frac{n}{2}\right)-1}(c, v_{2r+1}, v_{2r+2})\}$.

When i = n, $T_{i/2} = \{(v_1,v_n,c,v_3,v_4), (c,v_5)\}$.

Then $\{T_1,T_2,\ldots T_{n/2}\}$ is a directed out-tree cover for W_n and $\tau^+(W_n) \le \frac{n}{2}$. Thus, $\tau^+(W_n) = \frac{n}{2}$.

Theorem 2.6. For the symmetric wheel digraph,

$$\tau^+(W_n^*) = \begin{cases} n-1 & if\ n > 4 \\ 4 & if\ n = 4\,. \end{cases}$$

Proof: The symmetric wheel digraph W_n^* is shown in Fig. 5.

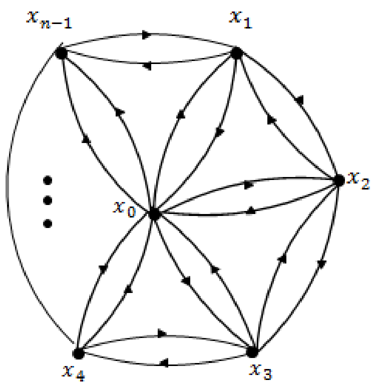

Fig. 5

Case (i): n>4.

By Theorem 2.3, we have $\tau^+(W_n^*) \ge n-1$.

Let $T_1 = (x_1,x_0) \cup (x_1, x_2, x_3, \ldots, x_{n-1})$,

$T_2 = (x_{n-1},x_0,x_1) \cup (x_{n-1},x_{n-2},\ldots,x_3,x_2)$

$T_3 = (x_2,x_0) \cup (x_{n-1},x_1) \cup \{(x_0,x_i)/ 3 \le i \le n-1\}$,

$T_4 = (x_3,x_0,x_2,x_1,x_{n-1})$

and for $4 \le i \le n-2$, $T_{i+1} = (x_i,x_0)$.

Then $\{T_1,T_2,\ldots,T_{n-1}\}$ is a directed out-tree cover for W_n^*, $\tau^+(W_n^*) \le n-1$. Thus, $\tau^+(W_n^*) = n-1$.

Case (ii): n = 4.

Let $T_1 = (x_1,x_0) \cup (x_1,x_2,x_3)$,

$T_2 = (x_3,x_0,x_1) \cup (x_3,x_2)$,

$T_3 = (x_2,x_0) \cup (x_3,x_1) \cup (x_0,x_3)$ and

$T_4 = (x_0,x_2,x_1,x_3)$.

Then $\{T_1,T_2,T_3,T_4\}$ is a directed out-tree cover for W_4^*, so that $\tau^+(W_4^*) \leq 4$. Since W_4^* is a 3-regular digraph, $\tau^+(W_4^*) \geq 3+1 = 4$, by Theorem 2.3.

Hence, $\tau^+(W_4^*) = 4$.

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